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ELIMINATION OF SKIDDING DUE TO STEERING MECHANISM ON MOTOR CARS

By A. L. CANDY, University of Nebraska.

When a wheel rolls along on the ground, if there is motion in the direction perpendicular to the plane of the wheel, then there is some sliding on the ground as well as a rolling forward in the plane of the wheel. This sliding is what I shall call "skidding." When a wheel rolls along a curved path, if the plane of the wheel is always tangent to its path, there is no skidding, but only a slight rotation about the point of contact with the ground, for the center of curvature of the path will then be on the projection of the axis of the wheel upon the ground.

In the ordinary wagon, or traction engine, the front axle turns on a single pivot at the center, so that the axes of the front wheels are always in the same straight line, viz., that of the front axle. If such a vehicle is set in motion, and the angle θ (Fig. 1) between the axles is kept constant, the four wheels, A, B, C, D, will describe concentric circles with center at O, the intersection of the lines of the axles, produced. Under these conditions there will be no skidding.

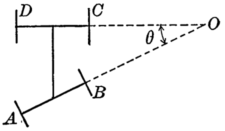

Fig. 1.

But in the automobile each front wheel turns on a separate pivot at the ends of the front axle. Hence the axes of the two front wheels are never in the same line, except when the car is moving straight forward. Let the two front wheels be turned through the same angle, and held rigidly. Then their axes, being parallel, will meet the line of the rear axle, produced, in two distinct points, O, and O' (Fig. 2), and make equal angles, θ, with it.

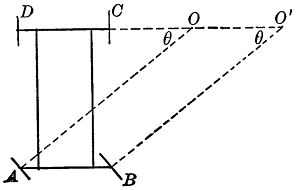

Fig. 2.

Now, if the car be made to turn about the point O as a center, the wheel B will skid. Likewise, if the car be made to turn about O' the wheel A will skid. If the car be made to turn about any other point on the line DC, both front wheels will skid. Under these conditions, if the car runs along any curved path whatever, there will be some skidding.

If, however, the inner wheel B is turned through a greater angle than the outer wheel A, so that the points O' and O coincide, then the car will run on a circle with O as center without any skidding whatever. Now in the steering gear of an automobile the cranks are not made parallel to the planes of the front wheels, so that the rod connecting the cranks is shorter than the distance between the two hinge joints. Hence, in the process of steering, the inner wheel is actually turned through a greater angle than the outer wheel.

The object of the following investigation is to determine whether by means of this simple scheme, all skidding can be practically eliminated. If so, what shall be the angle between the crank and the plane of the wheel?

Let $C(a, 0)$ and C' $(-a, 0)$ be the positions of the two pivot joints at the ends of the front axle (Fig. 3).

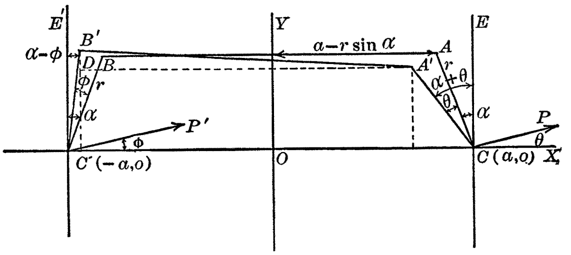

Fig. 3.

Let CE and $C'E'$ be perpendicular to $C'C$, and in the same plane with the cranks CA and $C'B$, of length r. Then CE and $C'E'$ will be horizontal diameters of the front wheels when the car is moving straight forward.

Let $\alpha = \angle ECA = \angle E'C'B$, the constant angles the cranks make with the planes of the wheels.

Let the inner crank CA be turned through an angle θ, to the position CA', then the outer crank $C'B$ will be turned through some angle φ, to the position $C'B'$, making

$$\angle\, ECA' = \alpha + \theta, \qquad \text{and} \qquad \angle\, E'C'B' = \alpha - \varphi.$$

It is required to find the locus of the intersection of the two lines CP and $C'P'$, which are perpendicular to the planes of the wheels, that is, the lines representing the axles of the wheels, produced.

Draw $A'D$ parallel to the x-axis $(C'OC)$, meeting the perpendicular from B' on the x-axis, in D. Then

$$A'D^2 + B'D^2 = A'B'^2, \qquad A'D = 2a - r\,[\sin(\alpha+\theta) + \sin(\alpha-\varphi)],$$

$$D\,B' = r\,[\cos(\alpha-\varphi) - \cos(\alpha+\theta)],$$

$$A'B' \equiv AB = 2(a - r\sin\alpha), \text{ a constant;}$$

$$\therefore [2a - r[\sin(\alpha+\theta) + \sin(\alpha-\varphi)]]^2 + r^2[\cos(\alpha-\varphi) - \cos(\alpha+\theta)]^2 = 4(a - r\sin\alpha)^2. \tag{1}$$

This equation reduces to the form

$$2a[\sin(\alpha+\theta) + \sin(\alpha-\varphi)] + r\cos(2\alpha+\theta-\varphi) = r\cos 2\alpha + 4a\sin\alpha, \tag{2}$$

from which the variables θ and φ must be eliminated.

The equations of the two lines CP and $C'P'$ are, respectively,

$$y = (x - a)\tan\theta, \qquad \text{and} \qquad y = (x + a)\tan\varphi,$$

whence

$$\tan\theta = \frac{y}{x-a}, \qquad \text{and} \qquad \tan\varphi = \frac{y}{x+a}.$$

Let $\lambda_1 \equiv (x - a)^2 + y^2$, and $\lambda_2 \equiv (x + a)^2 + y^2$. Then

$$\sin\theta = \frac{y}{\sqrt{\lambda_1}}, \qquad \cos\theta = \frac{x - a}{\sqrt{\lambda_1}}, \qquad \sin\varphi = \frac{y}{\sqrt{\lambda_2}}, \qquad \cos\varphi = \frac{x + a}{\sqrt{\lambda_2}},$$

$$\sin(\alpha + \theta) = \frac{(x - a)\sin\alpha + y\cos\alpha}{\sqrt{\lambda_1}}, \qquad \sin(\alpha - \varphi) = \frac{(x + a)\sin\alpha - y\cos\alpha}{\sqrt{\lambda_2}},$$

$$\sin(\theta - \varphi) = \frac{2ay}{\sqrt{\lambda_1\lambda_2}}, \qquad \cos(\theta - \varphi) = \frac{x^2 + y^2 - a^2}{\sqrt{\lambda_1\lambda_2}},$$

and

$$\cos(2\alpha + \theta - \varphi) = \cos 2\alpha \cos(\theta - \phi) - \sin 2\alpha \sin(\theta - \phi)$$
$$= \frac{(x^2 + y^2 - a^2)\cos 2\alpha - 2ay\sin 2\alpha}{\sqrt{\lambda_1\lambda_2}}.$$

Substituting these values in (2), and letting $s = \sin\alpha$, $c = \cos\alpha$, $s_2 = \sin 2\alpha$, $c_2 = \cos 2\alpha$, and $k = rc_2 + 4as$, we get

$$2a\left[\frac{s(x - a) + cy}{\sqrt{\lambda_1}} + \frac{s(x + a) - cy}{\sqrt{\lambda_2}}\right] + r\left[\frac{c_2(x^2 + y^2 - a^2) - 2as_2y}{\sqrt{\lambda_1\lambda_2}}\right] = k, \tag{3}$$

which is the equation of the locus.

Multiplying this equation by $\sqrt{\lambda_1\lambda_2}$ and rationalizing, we get

$$\begin{aligned}
\{&- 64a^4(s^2x^2 - c^2y^2 + as_2y - a^2s^2)^2 - 32a^2kr(s^2x^2 - c^2y^2 + as_2y \\
&- a^2s^2)[c_2(x^2 + y^2 - a^2) - 2as_2y] - 2k^2r^2[c_2(x^2 + y^2 - a^2) - 2as_2y]^2 \\
&- 8a^2k^2[\lambda_2(sx + cy - as)^2 + \lambda_1(sx - cy + as)^2]\}\lambda_1\lambda_2 + k^4\lambda_1^2\lambda_2^2 \\
&+ 16a^4[\lambda_2(sx + cy - as)^2 + \lambda_1(sx - cy + as)^2]^2 + r^4[c_2(x^2 + y^2 - a^2) \\
&- 2as_2y]^4 - 8a^2r^2[\lambda_2(sx + cy - as)^2 + \lambda_1(sx - cy \\
&+ as)^2][c_2(x^2 + y^2 - a^2) - 2as_2y]^2 = 0.
\end{aligned} \tag{4}$$

On expanding this equation, restoring the values of λ_1 and λ_2, which are both functions of x and y, and also restoring the value of $k \equiv rc_2 + 4as$, we obtain an equation of the 8th degree in which the coefficients of x^8, x^6, x^4, x^2, and the constant term all vanish. The equation can then be divided by $32a^2y$. Hence the x-axis may be regarded as a part of the locus.

Then, after an enormous amount of work in collecting and simplifying the twenty-five coefficients involved, we get the following equation (5) as the final equation of the locus:

$$
\begin{array}{llllllr}
2s^2 & [4a^2s^2 & +4arsc_2 & +r^2c_2{}^2 & & &]x^6y \\
+2s^2 & [4a^2(4s^2-1) & +12arsc_2 & +3r^2c_2{}^2 & & &]x^4y^3 \\
+2s^2 & [4a^2(5s^2-2) & +12arsc_2 & +3r^2c_2{}^2 & & &]x^2y^5 \\
+2s^2 & [4a^2(2s^2-1) & +4arsc_2 & +r^2c_2{}^2 & & &]y^7 \\
-2s_2 & [4a^3s^2 & +4a^2rs(1-3s^2) & +ar^2c_2(1-6s^2) & -r^3sc_2{}^2 & &]x^6 \\
+2s_2 & [4a^3(1-3s^2) & -4a^2rs(3-7s^2) & -2ar^2c_2(1-7s^2) & +3r^3sc_2{}^2 & &]x^4y^2 \\
+2s_2 & [4a^3(1-s^2) & -4a^2rs(3-5s^2) & -ar^2c_2(1-10s^2) & +3r^3sc_2{}^2 & &]x^2y^4 \\
+2s_2 & [4a^3s^2 & -4a^2rs(1-s^2) & +2ar^2c_2s^2 & +r^3sc_2{}^2 & &]y^6 \\
-2 & [4a^4s^2 & +4a^3rs(1-7s^2+6s^4) & +a^2r^2(1-16s^2+56s^4-44s^6) & +ar^3sc_2(1-2c_2{}^2) & -r^4s^2c^2c_2{}^2 &]x^4y \\
+2 & [4a^4(1-8s^2+10s^4) & -8a^3rs(1-9s^2+10s^4) & -a^2r^2(1-18s^2+80s^4-72s^6) & -2ar^3sc_2(1-2c_2{}^2) & +2r^4s^2c^2c_2{}^2 &]x^2y^3 \\
-2 & [12a^4s^2(1-2s^2) & +4a^3rs(1-11s^2+14s^4) & -2a^2r^2s^2(1-12s^2+14s^4) & +ar^3sc_2(1-2c_2{}^2) & -r^4s^2c^2c_2{}^2 &]y^5 \\
+6a^2s_2 & [4a^3s^2 & +4a^2rs(1-3s^2) & +ar^2c_2(1-6s^2) & -r^3sc_2{}^2 & &]x^4 \\
-as_2 & [8a^4 & -16a^3rs(1-s^2) & -2a^2r^2(3-20s^2+24s^4) & +4ar^3sc_2{}^2 & +r^4c_2(1-2c_2{}^2) &]x^2y^2 \\
+as_2 & [24a^4s^2 & -8a^3rs(1+s^2) & -4a^2r^2(1-6s^2+6s^4) & +2ar^3sc_2{}^2 & -r^4c_2(1-2c_2{}^2) &]y^4 \\
+2a^2 & [4a^4s^2(2-3s^2) & +4a^3rs(2-17s^2+18s^4) & +a^2r^2(2-35s^2+124s^4-100s^6) & +2ar^3sc_2(1-2c_2{}^2) & -2r^4s^2c^2c_2{}^2 &]x^2y \\
-\frac{1}{2}a^2 & [48a^4s^2(1-2s^2) & +16a^3rs(2-19s^2+22s^4) & +4a^2r^2(1-19s^2+84s^4-76s^6) & +8ar^3sc_2(1-2c_2{}^2) & -r^4(1-6s_2{}^2c_2{}^2) &]y^3 \\
-6a^4s_2 & [4a^3s^2 & +4a^2rs(1-3s^2) & +ar^2c_2(1-6s^2) & -r^3sc_2{}^2 & &]x^2 \\
+a^3s_2 & [24a^4s^2 & +8a^3rs(1-5s^2) & -2a^2r^2(1-2s^2-4s^4) & -2ar^3sc_2{}^2 & +r^4c_2(1-2c_2{}^2) &]y^2 \\
-2a^4 & [4a^4s^2(1-2s^2) & +4a^3rs(1-9s^2+10s^4) & +a^2r^2(1-18s^2+64s^4-52s^6) & +ar^3sc_2(1-2c_2{}^2) & -r^4s^2c^2c_2{}^2 &]y \\
+2a^6s_2 & [4a^3s^2 & +4a^2rs(1-3s^2) & +ar^2c_2(1-6s^2) & -r^3sc_2{}^2 & &]=0.
\end{array}
\qquad (5)
$$

Since y is a factor of the seventh degree terms in equation (5), the locus has an asymptote parallel to the x-axis. The equation of this asymptote is found by putting the sum of the x^6y and x^6 terms equal to zero.

This gives

$$y = 2\frac{c}{s}(a - rs) \equiv 2 \cot \alpha(a - r \sin \alpha). \tag{6}$$

Throughout the limited range of motion (variation of θ) that is possible in the automobile, the curve lies very close to this asymptote, and on the same side of it as the x-axis, which is in the position of the front axle.

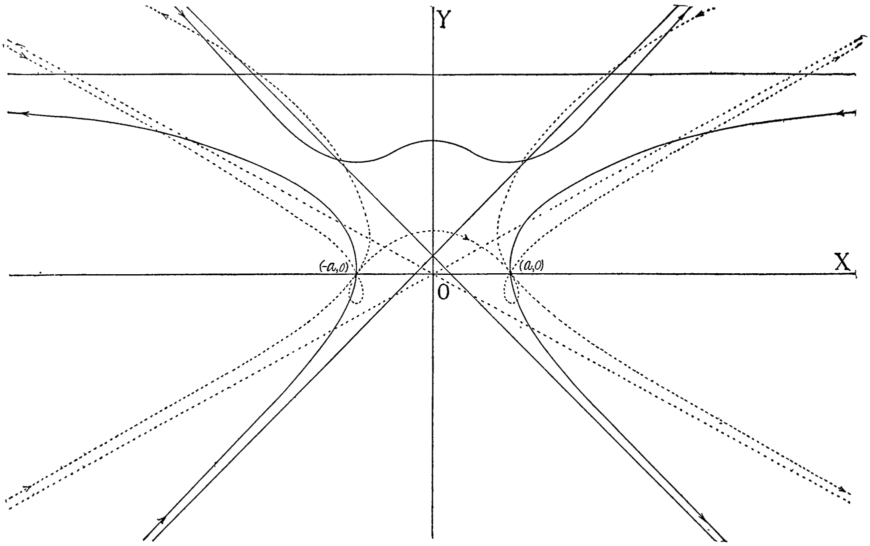

Fig. 4.

For continuous curve, $\alpha = 30°$; for dotted curve, $\alpha = 15°$.

Therefore, if, with given values of a and r, we find a value of α which makes the ordinate of this asymptote a little greater than the wheelbase of the car, there will be very little skidding due to the steering mechanism when the car runs on a curved path.

If $a = 2.1$ ft., $r = \frac{1}{2}$ ft., and $\alpha = 21°$, we get $y = 10.05$ ft.

These would be good values for a car with a wheel-base of 9.5 ft.

It is evident from equation (6) that y increases as α diminishes. Therefore, since a and r may be considered as practically the same in all cars, the greater the wheel-base the smaller the angle α must be.

So far it has been assumed that the cranks are behind the front axle. If the cranks project forward, as in some cars, they must then diverge from the planes of the wheels. In order to get the corresponding equations for this case, it will be sufficient to change the sign of r.

With $a = 2$ and $r = 1$ we get for this asymptote [Eq. (6)] the following equations:

When $\alpha = 30°$, $y = 3\sqrt{3}$. [See Fig. 4].
When $\alpha = 15°$, $y = 13-$. Not shown in Fig. 4.
When $\alpha = 45°$, $y = 2.6$. See dot and dash line in Fig. 5.
When $\alpha = 60°$, $y = 1.3$. See dotted line in Fig. 5.
When $\alpha = 90°$, $y = 0$.
When $\alpha \doteq 0$. $y \doteq \infty$.

If we let

$$l^2 = 4a^2s^2 + 4arsc_2 + r^2c_2^2 = (2as + rc_2)^2,$$

and

$$m^2 = c_2(4a^2 - 4ars - r^2c_2),$$

the 7th degree terms may then be written in the form

$$2s^2y(l^2x^2 - m^2y^2)(x^2 + y^2)^2.$$

Hence there is also a pair of asymptotes parallel to the two lines

$$l^2x^2 - m^2y^2 = 0.$$

After another long process of reduction we find the equations of these asymptotes to be

$$y = \pm\frac{l}{m}x + \frac{as(4a^2s^2 - 4ars^3 - r^2c_2)}{cm^2} \tag{7}$$

Let $a = 2$, and $r = 1$.[1] Then for this pair of asymptotes we get the following equations:

When $\alpha = 30°$, $y = \pm\dfrac{5}{\sqrt{23}}x + \dfrac{20\sqrt{3}}{69} = \pm x \tan 46°11.'5 + .5$. [See Fig. 4]

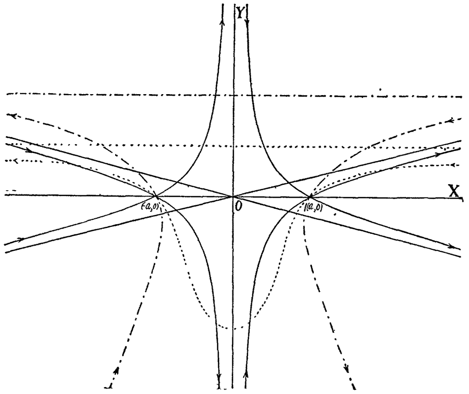

Fig. 5.
For continuous curve $\alpha = 0°$; for dot-dash curve, $\alpha = 45°$; for dotted curve, $\alpha = 60°$.

When $\alpha = 15°$, $y = \pm x \tan 29°29' + .03$. [See Fig. 4]

When $\alpha = 0$, $x^2 - 15y^2 = 0$. [See Fig. 5]

When $\alpha = 44°59'$,

$$y = \pm x \tan 88°53'.5 + 2948.$$

x-intercepts $= \pm 55.75$.

When $\alpha > 45°$, these asymptotes are imaginary, since m^2 contains the factor c_2, which is then negative.

As $\alpha \doteq 45°$, these asymptotes move off into the infinite region. Hence that part of the curve below the x-axis [Fig. 5] approaches in the limit the form of a parabola with infinite branches extending downward.

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[1] All the curves shown in Figs. 4 and 5, were constructed with $a = 2$ in. and $r = 1$ in. Hence the unit of the scale in these figures is half the distance from O to $(a, 0)$.

There are three rather distinct forms of the curve.

(1) When α is less than about 20°. This form is represented by the dotted curve in Fig. 4, which in the limit as $\alpha \doteq 0$, becomes the continuous curve in Fig. 5, the loops below the x-axis then becoming infinite and one part approaching coincidence with the x-axis.

(2) When $45° > \alpha > 20°$. The continuous curve of Fig. 4 is an example of this form, and as $\alpha \doteq 45°$ it becomes the dot and dash curve of Fig. 5, the upper branch in Fig. 4 then passing entirely into the infinite region.

(3) When $\alpha > 45°$. This form is seen in the dotted curve of Fig. 5, and it approaches coincidence with the x-axis as $\alpha \doteq 90°$.

When $\alpha = 0$ the equation of the curve is

$$y^2[4r^2x^4 - 4(4a^2 - r^2)x^2y^2 - 8a^2r^2x^2 + r^2(4a^2 - r^2)y^2 + 4a^4r^2] = 0. \tag{8}$$

When $\alpha = 45°$ we get

$$\begin{aligned}
&2x^2y(x^2 + y^2)^2 - 2(2a - r\sqrt{2})(x^2 - y^2)(x^2 + y^2)^2 \\
&\quad - (4a^2 - 4ar\sqrt{2} + 3r^2)x^4y - 2(2a^2 - 4ar\sqrt{2} + 3r^2)x^2y^3 \\
&\quad + (4ar\sqrt{2} - 3r^2)y^5 + 6a^2(2a - r\sqrt{2})x^4 - 2a(4a^2 - 2ar\sqrt{2} + r^2)x^2y^2 \\
&\quad + 2a(6a^2 - 3ar\sqrt{2} + r^2)y^4 + 2a^2(a^2 - 4ar\sqrt{2} + 3r^2)x^2y \\
&\quad + \tfrac{1}{2}(16a^3r\sqrt{2} - 12a^2r^2 + r^4)y^3 - 6a^4(2a - r\sqrt{2})x^2 \\
&\quad + 2a^3(6a^2 - 3ar\sqrt{2} + r^2)y^2 + a^4(4ar\sqrt{2} - 3r^2)y + 2a^6(2a - r\sqrt{2}) = 0.
\end{aligned} \tag{9}$$

When $\alpha = 90°$ we get

$$\begin{aligned}
&y[(2a - r)^2(x^2 + y^2)^3 - a(2a - r)^2(a + r)x^4 + a(2a - r)(6a^2 - 5ar \\
&\quad + 2r^2)x^2y^2 + a(12a^3 - 16a^2r + 6ar^2 - r^3)y^4 - a^3(2a - r)^2(a - 2r)x^2 \\
&\quad + \tfrac{1}{4}a^2(2a - r)(24a^3 - 28a^2r + 6ar^2 - r^3)y^2 + a^5(2a - r)^2(a - r)] = 0.
\end{aligned} \tag{10}$$

Equations (8), (9), and (10) were derived directly from equation (4), and also by substituting in equation (5), and the two results agreed.

The y-intercept of the curve is always $a \tan 2\alpha$.

When α is 30° and 60° these intercepts are $\pm a\sqrt{3}$, respectively. We find that the coördinates $(0, \pm a\sqrt{3})$ satisfy the corresponding equations. This serves as a check on the coefficients of the y terms.

If we let $y = 0$ in equation (5) we get $(x^2 - a^2)^3 = 0$, for all values of α. Hence the coördinates of the points $(\pm a, 0)$ always satisfy equations (5). When α is less than about 20°, the exact value depending upon the relative values of a and r, the curve goes through each of these points three times. These points are then real double points with respect to the "upper branch" of the locus. For larger values of α these points must be considered as conjugate or isolated points, with respect to this branch of the locus.

Thus these points are related to this branch of the locus in the same way as the origin is related to the conchoid $(x^2 + y^2)(x - a)^2 = b^2x^2$.

The coördinates (0, 0) always satisfy this equation, and when $y = 0$, we get $x = 0, 0, a \pm b$, but the curve does not cut the x-axis four times unless b $> a$. That is, the origin is a double point, or a conjugate point according as $b > a$, or $b < a$. The same thing is true in regard to the limaçon

$$(x^2 + y^2 - ax)^2 = b^2(x^2 + y^2).$$

NOTE ON THE INCENTERS OF A QUADRILATERAL.

By F. V. Morley, Johns Hopkins University.

The rectangular configuration of the incenters of an inscribed quadrilateral was known to Professor Neuberg of Liège before 1906.[1] It was rediscovered, as an isolated fact, by the writer in 1914, and several proofs were supplied by his father, Professor Morley. When the Monthly enlarged its scope in 1916 the proposition was sent in as a problem, appearing in March, 1917; the obvious extension and the writer's solution were published soon after.[2] In June, 1918, Professor Altshiller-Court published an article in this Monthly, rediscovering the configuration.[3]

The proposition as an isolated fact with reference to the inscribed quadrilateral was therefore worth discovery; but it will now be shown that the proposition appears in a well-known theory.

I.

There are several beautiful chains of theorems concerned with the elementary geometry of n-lines in a plane. One of these is Clifford's chain, and others of similar type are well known to students of metric or reflexive geometry. With slight modifications, they apply to directed as well as to undirected lines.[4]

One of these chains is concerned with the incenters of n-lines. In the present note it will be simpler to consider the lines as directed. For example, a triangle composed of three undirected lines has four incenters, using the term in its general meaning; but if the three lines are thought of as directed, there is only one circle tangent to all three in the proper sense, and hence only one incenter.

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[1] The reference here seems to be to Neuberg's article in *Mathesis*, 1906, pp. 14–17. But Neuberg published the result as a problem many years earlier (*Nouvelle correspondance mathématique*, Tome 1, 1875, p. 96), accompanied by the statement that it was extracted from *Archiv der Mathematik und Physik*, 1842, p. 328. Catalan's solution of the problem appears in *Nouvelle correspondance*, tome 1, pp. 198–200.—Editor.

[2] November, 1917, volume 24, pp. 429–430.

[3] Volume 25, 1918, pp. 241–246; for comment, see volume 26, 1919, pp. 65–66. Still more recently, the subject has been discussed in the comprehensive article by J. W. Clawson, in the *Annals of Math.*, vol. 20, p. 254 (1919). [Mr. Morley's paper was in the hands of the Editor some time before Professor Clawson's paper was published.—Editor.]

[4] Cf. F. Morley, *Transactions of the American Mathematical Society*, Vol. 1, pp. 97–115; and F. H. Loud, *Transactions of the American Mathematical Society*, Vol. 1, pp. 323–338.